www.grallator.co.uk
E-mail: chris@grallator.co.uk

Radioactive Particle Transport

One of the possible scenarios for a terrorist attack is the detonation of an explosive device that is encased in radioactive material; the so-called "dirty bomb". In this scenario fine particles of radioactive material are released into the environment which are transported by three mechanisms

- ballistics, particles are ejected in random directions and fall to earth under gravity.
- advection, i.e. being blown away by the wind.
- diffusion where very fine particles are dispersed as they travel.

The first mechanism is an important component in the determination of the so-called deposition velocity. This is the speed at which particles settle back to earth. The second and third mechanisms are the dominant method of transport for small particles. In the near-field and especially in urban environments, where buildings modify wind in a complex manner, advective transport can be complex. Over larger distance scales dispersion is often treated using Gaussian plume type approaches which include a simple wind average advective term in their formulation.

This note will develop some simple mathematical models based on the ballistics of the release to provide a background for some of the important aspects of radioactive particle dispersal. It is not intended to be a full study!

Ballistic behaviour

The ballistic behaviour of the particles can be modelled by assuming that at the scales of the particles Stokes flow is valid and that the friction on a spherical particle of radius r is given by

$$\underline{F} = -6\pi r\mu\underline{v} = -6\pi r\mu\frac{d\underline{x}}{dt}, \qquad (1)$$

where μ is the coefficient of viscosity for air and v is the velocity of the sphere. If the velocity of the particle is resolved into horizontal and vertical components the following equations of motion can be constructed from Newton's Second Law.

$$\begin{aligned} H: \quad & m\frac{d^2x}{dt^t} = -6\pi r\mu\frac{dx}{dt} \\ V: \quad & m\frac{d^2y}{dt^t} = -6\pi r\mu\frac{dy}{dt} - mg \end{aligned} \qquad (2)$$

where x and y are the horizontal and vertical displacements from an origin, m is the particle mass and g is the acceleration due to gravity. The coordinates are such that right and up are in the positive direction.

For a spherical particle of density ρ, the mass is given by

$$m = \frac{4}{3}\rho\pi r^3 \qquad (3)$$

Substituting this into (2) yields

$$\begin{aligned} H: &\quad \frac{d^2x}{dt^t} + \alpha \frac{dx}{dt} = 0 \\ V: &\quad \frac{d^2y}{dt^t} + \alpha \frac{dy}{dt} = -g \end{aligned} \tag{4}$$

where

$$\alpha = \frac{9\mu}{2\rho r^2}\,. \tag{5}$$

The parameter α gives the relative importance of viscous drag effects to inertial effects. Solving (4) is reasonably straightforward using complementary functions with particular integrals, substitution of $p = \frac{dx}{dt}, q = \frac{dy}{dt}$ and integrating twice, or by Laplace transforms.

For the first method the auxiliary equation is the same for both the horizontal and the vertical and leads to

$$m^2 + \alpha m = 0 \quad \Rightarrow \quad m = 0 \quad \text{or} \quad m = -\alpha \tag{6}$$

so that the complementary functions are

$$\begin{aligned} H: &\quad x = A_x + B_x e^{-\alpha t} \\ V: &\quad y = A_y + B_y e^{-\alpha t} \end{aligned} \tag{7}$$

The horizontal equation in (4) is homogeneous so the particular integral is zero. The full solution is obtained by considering the initial conditions for position and speed:

$$\begin{aligned} & x(t=0) = 0 \\ & \left.\frac{dx}{dt}\right|_{t=0} = U_0 \end{aligned} \tag{8}$$

Substituting into the Horizontal expression in (7), along with t=0, leads to

$$\begin{aligned} & 0 = A_x + B_x e^0 \quad \Rightarrow \quad A_x = -B_x \\ & U_0 = -\alpha B_y e^0 \quad \Rightarrow \quad B_x = -\frac{U_0}{\alpha}\,, \end{aligned} \tag{9}$$

so that

$$x = \frac{U_0}{\alpha}\left(1 - e^{-\alpha t}\right). \tag{10}$$

The maximum horizontal distance that the particle can travel before coming to rest due to friction is x_m, given by

$$x_m = \frac{U_0}{\alpha}\,. \tag{11}$$

www.grallator.co.uk
E-mail: chris@grallator.co.uk

For the vertical component some care must be taken with the initial conditions as both the initial speed and acceleration must be correct. This leads to

$$\begin{aligned} & y(t=0)=0 \\ & \left.\frac{dy}{dt}\right|_{t=0}=V_0 \\ & \left.\frac{d^2y}{dt^2}\right|_{t=0}=-\alpha V_0-g \end{aligned} \tag{12}$$

Substituting the first and last condition into the Vertical expression in (7), along with t=0, leads to

$$\begin{aligned} & 0=A_y+B_y e^0 \quad \Rightarrow \quad A_y=-B_y \\ & -\alpha V_0-g=\alpha^2 B_y e^0 \quad \Rightarrow \quad B_y=-\frac{(g+\alpha V_0)}{\alpha^2}, \end{aligned} \tag{13}$$

so that the complementary function is

$$y_{CF}=\left(\frac{g+\alpha V_0}{\alpha^2}\right)\left(1-e^{-\alpha t}\right). \tag{14}$$

The particular integral is found by inspection of the equation

$$\frac{d^2y}{dt^t}+\alpha\frac{dy}{dt}=-g, \tag{15}$$

which is true if

$$y_{PI}=-\frac{gt}{\alpha}, \tag{16}$$

The general solution for y is then

$$y=\left(\frac{g+\alpha V_0}{\alpha^2}\right)\left(1-e^{-\alpha t}\right)-\frac{gt}{\alpha}. \tag{17}$$

Note that if the initial y-velocity rather than acceleration were used in determining A_y and B_y then the second derivative of (17) w.r.t. time would be incorrect as the acceleration due to gravity term would be missing.

There is no maximum vertical distance; equation (17) states that after a transient period the particle will fall with a constant terminal velocity of g/α. This is known as the gravity deposition velocity.

Are the solutions sensible?

Expressions (10) and (17) give the evolution of the x and y coordinate as a function of time and they can be shown to be solutions to the differential equations (4). It is worth a few moments reflection to reassure ourselves that they are correct.

The difference between the horizontal and the vertical equation of motion is that the vertical motion is subject to acceleration due to gravity. One can ask then, if $g \to 0$ in (17), is the resulting expression consistent with (10). Simple substitution of $g = 0$ in (17) yields

$$y = \frac{V_0}{\alpha}\left(1 - e^{-\alpha t}\right), \tag{18}$$

which is consistent with (10).

Another question that may be asked is what happens when there is no viscosity? In this case $\alpha = 0$ and it would appear that both (10) and (17) are not defined when, in fact, we would expect the standard ballistics equations without friction to be recovered. The behaviour of the expressions as $\alpha \to 0$ can be analysed by using l'Hospital's rule. For expression (10):

$$x = \frac{U_0}{\alpha}\left(1 - e^{-\alpha t}\right) = \frac{f(\alpha)}{g(\alpha)}, \tag{19}$$

where $f(\alpha)$, $g(\alpha) \to 0$ as $\alpha \to 0$. The limit is given by

$$\lim_{\alpha\to 0}\frac{f(\alpha)}{g(\alpha)} = \frac{U_0\left(1 - e^{-\alpha t}\right)}{\alpha} = \lim_{\alpha\to 0}\frac{f'(\alpha)}{g'(\alpha)} = \frac{U_0 t e^{-\alpha t}}{1} \to U_0 t \quad \text{as} \quad \alpha \to 0. \tag{20}$$

This recovers the standard result of uniform motion in a straight line for $\alpha \to 0$. For the vertical expression some rearrangement is required and the second derivative must be used in l'Hospital's rule:

$$y = \left(\frac{g + \alpha V_0}{\alpha^2}\right)\left(1 - e^{-\alpha t}\right) - \frac{gt}{\alpha} = \frac{\left(g + \alpha V_0\right)\left(1 - e^{-\alpha t}\right) - \alpha g t}{\alpha^2} = \frac{f(\alpha)}{g(\alpha)}. \tag{21}$$

where $f(\alpha)$, $g(\alpha) \to 0$ as $\alpha \to 0$, allowing application of the rule.

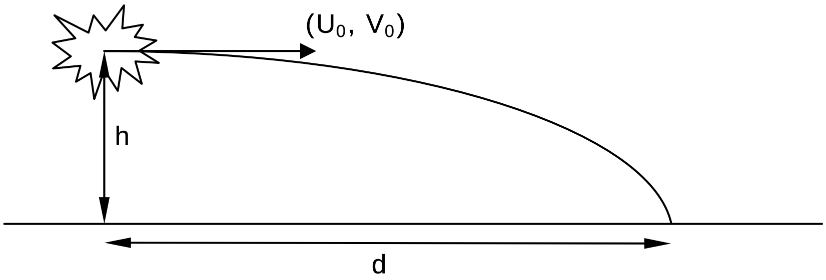
$$\begin{aligned}\lim_{\alpha\to 0}\frac{f(\alpha)}{g(\alpha)} &= \frac{\left(g + \alpha V_0\right)\left(1 - e^{-\alpha t}\right) - \alpha g t}{\alpha^2} \\ &= \lim_{\alpha\to 0}\frac{f''(\alpha)}{g''(\alpha)} = \frac{2V_0 t e^{-\alpha t} - \left(g + \alpha V_0\right)t^2 e^{-\alpha t}}{2} \quad \to V_0 t - \tfrac{1}{2}gt^2 \quad \text{as} \quad \alpha \to 0\end{aligned} \tag{22}$$

Again the standard equation for acceleration under gravity is recovered for $\alpha \to 0$. These limits can be checked on a spreadsheet. However, be careful as numerical rounding errors will cause a divergence error for α smaller than 10^{-12} for equation (10) and 10^{-6} for equation (17). **Always make sure the results given by a computer calculation are sensible. Small parameters and exponential functions often highlight numerical errors caused by a computer's fixed precision arithmetic.**

www.grallator.co.uk
E-mail: chris@grallator.co.uk

Investigation of Ballistic behaviour – no wind

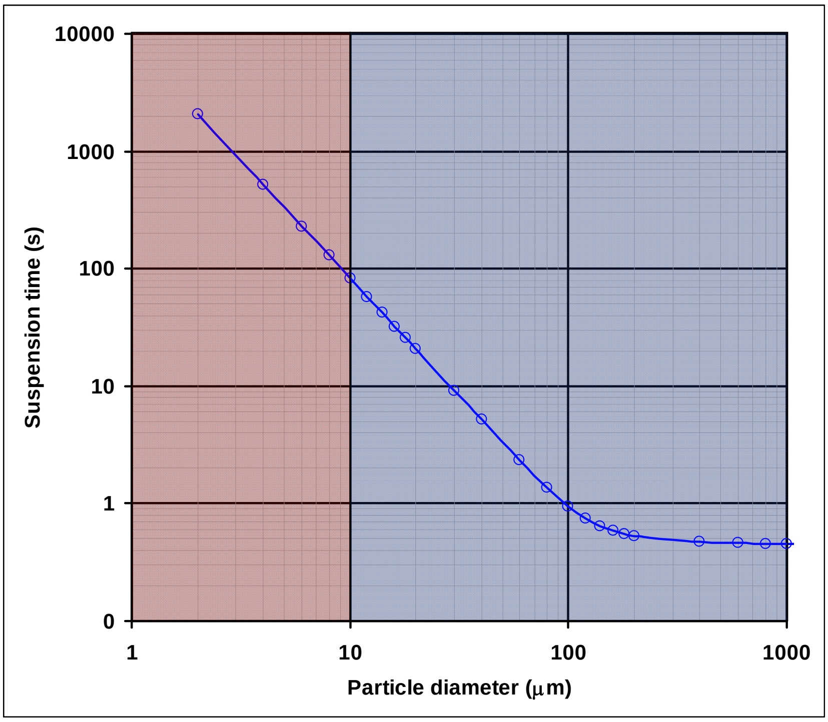
Consider the case of a particle of radius r (diameter 2r) released at a height h with initial horizontal and vertical components of velocity of U_0 and V_0. How far will the particle travel, d, before it hits the ground?



To study this we'll assume that $V_0 = 0$, i.e. particles are initially released travelling parallel with the ground. In reality, some particles will go up initially and go further, while others will go down and travel less, but for comparative purposes we'll consider the ground-parallel fraction.

To find d, equation (17) is solved for t when y = -h (ground is below the source and down is negative in this model). This gives the suspension time; the time that the particle is in the air. Substituting this time into (10) allows gives d.

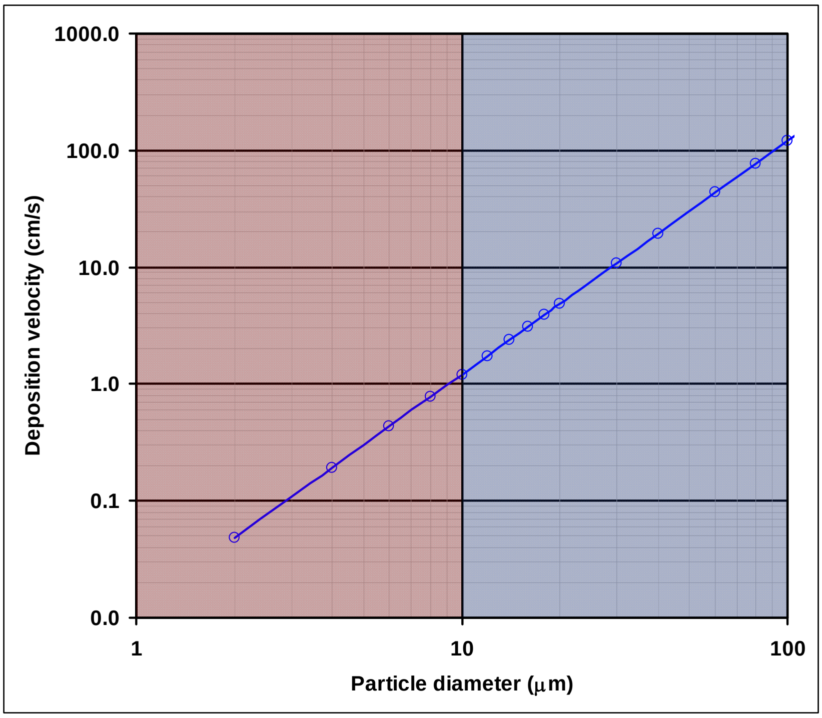
Assuming h=1 m, V_0=0, g=9.8 ms^{-1}, $\mu=1.8\times10^{-5}$ Nsm^{-2}, $\rho=4\times10^{3}$ kgm^{-3} and that air-buoyancy can be neglected, the suspension time for different particle diameters is shown below.



Note, the formulation above uses radii while the results are presented using diameter values. In this figure the particles in the red region are those that are respirable, while those in the blue region are not. All particles will contribute to external dose but only the respirable ones will contribute to internal dose. Note that above 100 μm the graph flattens. This is the point at which air-drag is unable to effectively slow the particles fall and a constant time to fall is achieved. Below this the air-drag slows the fall in inverse proportion to the particle size so that, for example, while a 100 μm particle takes about 1 second to fall 1m, while a 10 μm particle takes about 80 seconds.

In planning scenarios it has been assumed[1] that about 90% of the original source may be liberated during the explosion and that particles will range in size from 1 to 150 μm, with 100 μm being the most probable size. From the above plot the smallest particles can remain suspended for several tens of minutes while the most probable size settles in about 1 second. Larger particles settle slightly quicker than this.

The suspension time graph can be used to plot the approximate deposition velocity as shown below.



The HotSpot code uses average values of 0.3 cm/s for respirable particles and 8cm/s for non-respirable particles, which corresponds to ~5 μm and ~20 μm diameter representative particles respectively. Note, that this is based on simple ballistics only. Advection/diffusion and other effects are ignored.

The ballistic range, obtained by substituting the deposition time and U_0 into (10), is plotted below.

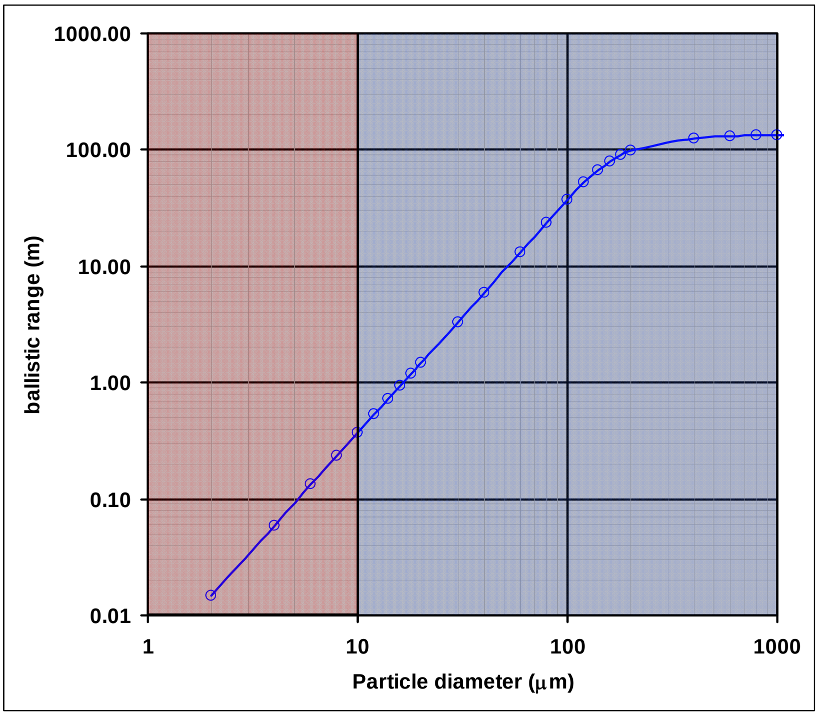
Note how viscous air drag slows the small particles down rapidly so that they do not travel an appreciable distance from the source. (NOTE, the above is an idealised situation where the source is a point source and air-expansion, increased buoyancy and turbulent motion due to the explosion is neglected.) Larger particles travel further up to the limit of ballistic behaviour with no appreciable air-drag - the region where the curve flattens. This graph gives a first-

[1] Homeland Security Planning Scenarios Executive Summary, see RDD section.

www.grallator.co.uk
E-mail: chris@grallator.co.uk

order estimate of the initial spread of material due to the explosion (within the large approximation of neglecting the explosion itself!).

For a planning scenario with the most probably particle size being 100 mm, the initial dispersal range is approximately 40m, while respirable particles remain reasonable close to the device as viscous drag slows them to ambient speeds (zero in this case) very effectively.



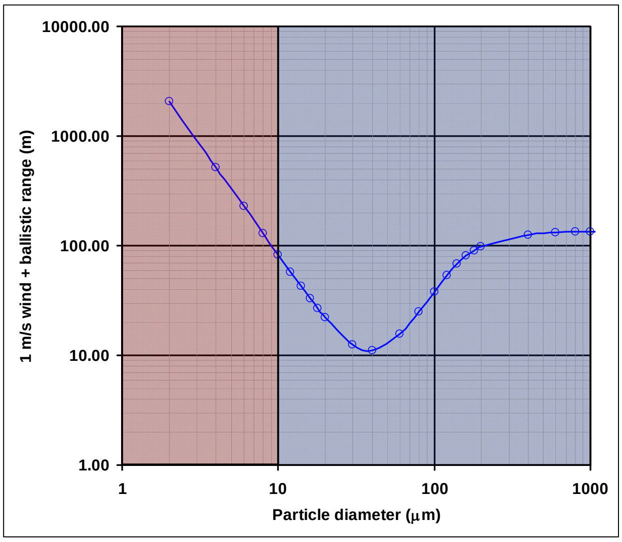
No wind

With Wind

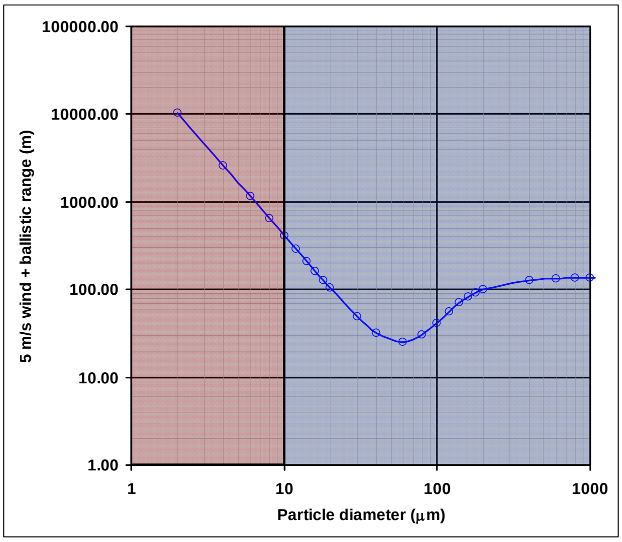
The above shows a reasonably localised spread of material. Consider now the case of a 1 ms^{-1} wind blowing uniformly that is able to advect the particles while they are suspended. In this case the long-time suspension of small particles allows the wind to carry them considerable distances before settling. In the 1 ms^{-1} wind example shown distances from 80 – 2000 m are achieved for respirable particles. Notice how the graph also predicts a particle size that will travel the least distance. This is due to it being small enough for air drag to slow it down but not so small that this suspends it for the long time periods required for significant advective transport. Above about 100 μm the advective and ballistic behaviours are comparable. Increasing the wind speed increases the potential transport distance as shown in the 5 ms^{-1} example.

Note on large-scale distances lateral spread will be significant. This is accounted for in models such as the Gaussian plume approach.

www.grallator.co.uk
E-mail: chris@grallator.co.uk



$1ms^{-1}$ wind



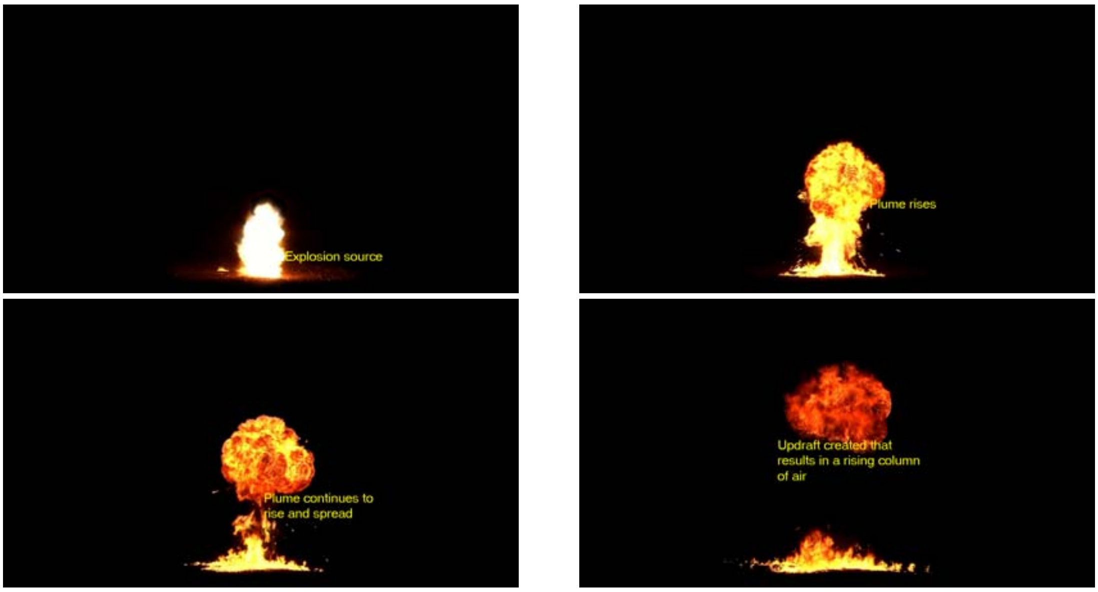
$5ms^{-1}$ wind

What's missing?

Lots!

The previous has been a "back-of-the-envelope" approach but is useful for gaining insight into some of the important characteristics of a release. A large approximation has been made by assuming the particles are released horizontally from the source with a given initial speed. This speed will be dependent on the explosion size which will determine the initial cloud size that subsequently settles. Clearly the analysis should cover all possible angles of release.

Another gross simplification in the above is the neglect of fluid-dynamics and the buoyant plume created by the explosion. An example of the behaviour is shown below and its effect would be to increase the initial spread of particles and to provide significant lift so that dispersal will be over a wider area.



This introduces the next gross simplification in this note; the detailed effects of the air in motion. This has been introduced in the form of a simple wind-drift model and has also discussed the effect of a buoyant plume. Both of these relate to the air as a fluid in motion and in the near field, especially in an urban environment, this flow is vital for determining details of dispersal. Buildings tend to produce complex air-flows inducing turbulent motion that increases the potential for dispersal. They can also produce eddies that concentrate particles in “stagnant” points. Note all this is a 3d effect so that such eddies can be produced not only by wind blowing round the side of a building but also over the top which can lift particles to higher altitudes. (Note, the above has neglected buildings altogether!)

Further details such as the location of the atmospheric temperature inversion layer are also important as this effectively caps particles rising too high.

Another detail that is missing is re-suspension. It has been assumed that once a particle strikes the ground it remains where it is. In fact there is the possibility that particles may be re-suspended in the air and carried further.

Chris Robbins, June 2009.

For more details of Grallator Modelling, Simulation and Software Services contact

Chris Robbins
Email: chris@grallator.co.uk
Tel. : 01663 749785
www.grallator.co.uk